Spectral decay of the sinc kernel operator and approximation by Prolate Spheroidal Wave Functions.

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The purpose of this talk is to give a summary of some of the recent results obtained in collaboration with A. Bonami. These results deal with the accurate explicit estimates of the PSWFs and their associated eigenvalues. Note that for a fixed real number $c > 0$, called the bandwidth, the prolate spheroidal wave functions (PSWFs) $(\psi_{n,c})_{n\geq 0}$ have been largely studied and used in various classical as well as in some emerging applications in different scientific areas such as Signal Processing and Physics. The PSWFs were first known as the bounded eigenfunctions of the Sturm-Liouville operator's L_c defined on $C^2([-1,1])$ by $L_c(\psi) = -(1-x^2)\frac{d^2\psi}{d\,x^2} + 2x\frac{d\psi}{d\,x} + c^2x^2\psi$. A breakthrough in the subject of the PSWFs is due to D. Slepian, H. Landau and H. Pollack, who have shown that the $\psi_{n,c}$ are also the eigenfunctions of the integral operators F_c and $Q_c = \frac{c}{2\pi}F_c^*F_c$, defined on $L^2([-1,1])$ by

$$F_c(f)(x) = \int_{-1}^{1} e^{i\,c\,x\,y} f(y)\,dy, \quad Q_c(\psi)(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\,\psi(y)\,dy. \qquad (1)$$

The PSWFs exhibit the desirable properties to form an orthogonal basis of $L^2([-1,1])$, and an orthonormal basis of the Paley-Wiener space $B_c = \{f \in L^2(\mathbf{R}),\ \text{Support}\ \widehat{f} \subset [-c,c]\}$. We let $(\lambda_n(c))_n$ denotes the infinite sequence of the eigenvalues of Q_c, arranged in the decreasing order. Many of the applications of the PSWFs heavily rely on their explicit analytic properties as well as on the precise behaviour and decay rate of the corresponding eigenvalues $(\lambda_n(c))_{n\geq 0}$. Although, there exists a rich literature on the numerical computation and asymptotic behaviour of the $\psi_{n,c}$ and $\lambda_n(c)$, very little is known about explicit estimates.

In this talk, we first give new bounds of the $\psi_{n,c}$ and the eigenvalues $\chi_n(c)$ of the differential operator L_c. Then, under a condition given in term of the quantity $q = c^2/\chi_n(c)$, we prove that $\psi_{n,c}$ is uniformly approximated on $[-1, 1]$ by a single function involving the function J_0 : the Bessel function of first kind and order zero. As a consequence of this uniform approximation, we provide a sharp exponential decay rate of the $\lambda_n(c)$ which is given by

$$\widetilde{\lambda}_n(c) = \frac{1}{2} \exp\left(-\frac{\pi^2}{2} n \int_{\Phi\left(\frac{2c}{\pi n}\right)}^{1} \frac{1}{t(\mathbf{E}(t))^2}\, dt \right). \tag{2}$$

Here, $\mathbf{E}(x) = \displaystyle\int_0^1 \sqrt{\frac{1 - x^2t^2}{1 - t^2}}\, dt$ is the elliptic Legendre integral of the second kind and $\Phi = \Psi^{-1}$, where $\Psi(x) = \dfrac{x}{\mathbf{E}(x)}$, $0 \le x \le 1$. As an application, we give the quality of approximation by the PSWFs in the spaces of band-limited, almost band-limited functions as well as on the Sobolev spaces $H^s([-1, 1]), s > 0$. The different results of this talk, are illustrated by some numerical examples.