Particle systems as solutions of SDEs systems

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Consider the following system of SDEs

$$d\lambda_i = \sigma_i(\lambda_i)dB_i + \left(b_i(\lambda_i) + \sum_{j\neq i} \frac{H_{ij}(\lambda_i,\lambda_j)}{\lambda_i-\lambda_j}\right) dt, \quad i=1,\dots,p, \tag{0.1}$$

describing ordered particles $\lambda_1(t) \le \ldots \le \lambda_p(t)$, $t \ge 0$ on $\mathbf{R}$. Here B_i denotes a collection of one-dimensional independent Brownian motions.

The SDEs systems (**??**) contain the following systems

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)dB_i + \left(b(\lambda_i) + \sum_{j\neq i} \frac{G(\lambda_i,\lambda_j)}{\lambda_i-\lambda_j}\right) dt, \quad i=1,\dots,p,$$

where $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, $\beta > 0$ and $g,h,b : \mathbf{R} \to \mathbf{R}$. It was shown in [**?**] that for the starting point having no collisions this system describes the eigenvalues of the S_p-valued process X_t satisfying the following matrix valued stochastic differential equation

$$dX_t = g(X_t)dW_t h(X_t) + h(X_t)dW_t^T g(X_t) + b(X_t)dt,$$

where the functions g,h,b act spectrally on S_p, the vector space of symmetric real $p \times p$ matrices and W_t is a Brownian matrix of dimension $p\times p$. Thus the systems (**??**) contain Dyson Brownian Motions, Squared Bessel particle systems, Jacobi particle systems, their β-versions and other particle systems crucial in mathematical physics and physical statistics([**?**],[**?**]).

Note that the functions $\frac{H_{ij}(\lambda_i,\lambda_j)}{\lambda_i-\lambda_j}$ describe the repulsive forces with which the particle λ_i acts on the particle λ_j. On the other hand the singularities $\frac{1}{\lambda_i-\lambda_j}$ make the SDEs system (**??**) difficult to solve, especially when the starting point $\Lambda(0)$ has a collision $\lambda_i(0) = \lambda_j(0)$. The most degenerate case $\lambda_1(0) = \ldots = \lambda_p(0)$ is of great importance in applications.

In some particular cases (Dyson Brownian Motions, some Squared Bessel particle systems), the existence of strong solutions of (**??**) has been established by Cépa and Lépingle, using the technique of Multivalued SDEs([**?**], [**?**]).

We prove the existence of strong and pathwise unique non-colliding solutions of (**??**), with a degenerate colliding initial point $\Lambda(0)$ in the whole generality, under natural assumptions on the coefficients of the equations in (**??**). Our approach is based on the classical Itô calculus, applied to elementary symmetric polynomials in p variables $X = (x_1,\dots,x_p)$

$$e_n(X) = \sum_{i_1<\ldots<i_n} x_{i_1}x_{i_2}\ldots x_{i_n},$$

as well as to symmetric polynomials of squares of differences between particles

$$V_n = e_n(A), \quad \text{where } A = \{a_{ij} = (\lambda_i-\lambda_j)^2 : 1 \le i < j \le p\}.$$

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